Embedding Minkowski space and Maxwell's equations of vacuum in a 5-dimensional Riemannian space

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Abstract

About 150 years ago, in summer 1854, Berhard Riemann held his habilitation lecture to Carl Friedrich Gauss in Göttingen to become a professor and built up a scientific basis to combine Minkowski space and electromagnetism, called Riemannian Geometry. At the same time Maxwell began his program of electromagnetism, Riemann had developed a wave-equation for the electric potential.

Riemann's geometry is also the basis for general relativity, which we neither need here nor want to discuss here.

To carry out this embedding two things besides Riemann's geometry are needed: firstly Michelson's and Morley's Experiment (1881) and its interpretation by special relativity for including also indefinite metrics to define Minkowski space and secondly Kaluza's Ansatz to place the differential operators of Maxwell's equation in Christoffel symbols of a curvature tensor. The embedding is presented here strictly by use of only differential geometry (curvature tensor) with indefinite metric and Maxwell's equations, without knowledge from or use of General Relativity. As a result electromagnetic energy shows to be a source for curvature, a hint to General Relativity, but on a very different path than Einstein has gone.

Keywords: electromagnetism – special relativitiy – Minkowski space – metric tensor – Riemannian geometry – Kaluza theory – scalar waves

1 Introduction

Riemann's geometry is a very general concept, to which Euclidean, Gaussian, Bolyai's and Lobachevski's geometry can easily be acommodated. But his geometry is not a generalization of those geometries, because it builds up on a very different basis. Euclid started with points, lines and angles in plane wheras Riemann uses n-tuples from manifolds (coordinate systems) extended by a distance function defined between coordinates giving a metric.

Embedding Minkowski space and Maxwell's equations in one 5 dimensional Riemannian space is pure mathematics and at best an asymptotic limit of a physical model for small fields. The intention for doing the embedding even so is didactical: to point out the idea behind it and to extend standard literature of classical electrodynamics [1] by a link to *Kaluza* theories. That the embedding is successful is surprising and known as a miracle found by Kaluza.

With his theory Kaluza unified gravitation with electromagnetism and founded the concept force = curvature of space [2], but here we exclude General Relativity by using only Minkowski metric.

Seventy years before Kaluza, Riemann had pursued a similar trace, but without success. He died early at the age of 40 from pneumonia. In his physical speculations from March 1st 1853 he examined deformations of a hypothetical liquid matter in a 3D Euclidean space. He was led to quadratic differtial forms, which coefficients were dependent not only from the 3 coordinates of space but also of time. He got a bunch of Riemannian metrics in 3 spatial dimensions, which he wanted to relate to propagation of gravity, light, and heat radiation, the basic idea for unifying forces [3].

As Dedekind cited from a letter dated December 28th 1854, Riemann continued his research about connections between electricity, galvanism, light and gravity immediately after his habilitation lecture. He wrote he easily could publish a paper about the subject and that he had reason to believe that Gauss too had worked on the same subject for several years.

But he never published such a paper. In a letter to his brother, Riemann reported June 26th 1854 that he dived so deep into his work on the principles of nature that he could not get over it even during the preparation of habilitation lecture, in which he closed with the hint "The reason for a metric in the space we live in has to be searched in forces acting upon this space" [4].

2 Discovery with scientific methods

We can believe Riemann's speculations or not, but what do we really know? Science obtains its knowledge and results by observation, experiment and mathematical descripton. Measurements and results build up first-hand a disconnected collection of descriptions, from which one tries to derive a more general context or a law or a principle. Several partial laws derived from a huge base of observations are combined to form a theory. This so called *inductive method* nowadays is the basic method for researchers and research in science.

Those basic laws found via the inductive method generalize the results of the experiments. Through generalization the content of those basic laws is more than the underlying facts. These basic laws allow us to derive estimated results for planned experiments (*deductive method*), similar to the methods of geometry. That those achieved new laws are correct can only be proved by repeated experiments and their accordance with prediction.

A priori it is uncertain, whether such a deductive treatment in physics is possible. It is *still a surprising fact*, that the results of experiments in the past, the present and the future are determined by a few laws, otherwise results would be an arbitrarily collection of disconnected facts [5].

The static laws of the three forces electricity, magnetism and gravitation were found easily as:

$$F_{el} = \frac{1}{4\pi\epsilon_0}\frac{q_1\ q_2}{r^2} \qquad F_{mag} = \frac{\mu_0}{4\pi}\frac{\vec{m}_1 \cdot \vec{m}_2}{r^2} \qquad F_{grav} = \gamma\frac{M_1\ M_2}{r^2}$$

All three consist of a product of the strengh of the two sources, multiplied by the inverse square of the distance between the sources. This similarity at once led to the assumption that there is some kind of link between them. Faraday made experiments with electric charges in a gravitational field in the shot tower of the houses of parliament, but could not detect any deviation [6]. He did not find a way to unify electricity with gravity.

A new development started, when Oerstedt found in 1820 that moving charges carry a magnetic field with them. The description of dynamical laws was much more difficult than those for the static laws. Interim results even seemed to violate the principle of energy conservation. As is well known, Maxwell combined the electric and magnetic field to one electromagnetic field, but with much more complicated laws than for static fields. Where does this complexity with gradient, rotation and vector potential come from ?

Primarily starting from Cavendish's and Faraday's experimental results and definitions Maxwell wrote down all knowledge about electricity of those days in his book "A treatise on Electricity an Magnetism" in 1873. With his fundamental equations Maxwell included all experimental results of his past, but also those of the following 20 years. It was possible to derive a wave equation from Maxwell's equations, which was proved by Heinrich Hertz in 1888 [7]. Since then a bunch of phenomena could be combined, because they have the same origin, namely: electricity, magnetism, radio, micro waves, heat radiation, light, UV, X-rays.

Historically classical electrodynamics became a *compact* and *fully deductively* treatable domain. The whole electrodynamics can be handled deductively on the basis of Maxwell's equations: "Maxwell's theory is the system of Maxwell's equations", Heinrich Hertz said [8]. A discussion of these basic equations could lead to an understanding of the phenomenon they describe. A successful concept was found in the explanation by the mechanics of particle systems, as for example the kinetic gas theory. There were several attempts to apply this concept also to Maxwell's equations, but all have failed (using particles with rest mass). That Maxwell's equations can not be explained by the mechanics of particle systems was a result of long lasting historical process. The searched concept was not found.

Einstein's description of gravity was a curvature of spacetime (1915). By extending this concept with an additional dimension, Kaluza was able to place Maxwell's equation into the curvature of a 5 dimensional space. So this is the concept searched for to join the three static force laws that were then known. It has been extended by more dimensions for string theory and others [9].

As we will see later in the text, the Riemannian curvature tensor has the neccessary complexity to explain the complicated dynamic laws of moving charges and at the same time is a simple concept, just curvature of space, so there is no need to use Occam's razor on this theory.

Before we take a look on Maxwell's equations from a new view, the basics shall be recapulated here by the questions 1) What do we understand under Maxwell's equations nowadays ? 2) What is the phenomenon behind them ? and 3) How can they be placed into a curvature tensor ?

3 Maxwell's equations nowadays

A key for understanding electromagnetism was Faraday's idea of electric and magnetic *lines of forces* (field lines), which should surround a charge or a magnet and be extended over the whole space. Test charges or test magnets are pulled along the course of these lines from one end to the other. The usage of such test bodies is a way to trace these lines of forces. They should exist in space also when there are no test charges or test magnets present in space on which they could act. This way the lines of forces determine a force for every point of space which gives the definition of a *force field*.

In Faraday's *field theory* with a *short-range* action an iron filing interacts with the lines of forces in its close proximity and not with the distant magnet. In contrast to this Newton's law of gravity is a *long-range action theory* where a force grasps directly from a mass point over a distance to the center of another mass without saying anything about how these both masses are connected over the distance or what is happening in the space between these two mass points.

A force field can be represented by assigning an arrow to each point in space for each moment in time. The direction of the arrow corresponds to the direction of the force and the length of the arrow corresponds to the strengh of the force. The mathematically exact term for this is a time-dependent *vector field*.

In today's form of Maxwell's equations as developed by Hertz and Heaviside this occurs as two time-dependent vector fields: the *vector field* E for the electric force field and the *vector field* B for the magnetic force field.

In a vector field there are points in space at which many lines of force start and diverge from (*source*) and other points in which they join up and end (*drain*). There are also lines of force which run from a point leading through an area of space and then turn back to this point again, they follow a ringlike shaped course (*rotational field*). Also for this there is a mathematically exact description in vector analysis: the operators *divergence* (div) and *rotation* (curl or rot), these both also occur in Maxwell's equations.

The *divergence operator* yields a positive value at points in space where sources are located (positive charge), in accordance a negative value at points where drains are located (negative charge) and at all other points, where the lines of force only pass through, it yields zero (empty space). It follows that a pure rotational field, in which there is neither a starting point nor an end point, the divergence is zero everywhere (magnetic field without magnetic charges).

To describe the *rotation operator* one has to change to hydrodynamics, where vector fields represent the velocities of flowing liquids (e.g. water). Putting a cork on the surface of such a streaming liquid, the rotation operator, applied to the speed field, describes the rotation of the cork on the liquid around its own axis. Only if the cork does rotate, the rotation operator yields a value different from zero. In this point the speed field has a *curl*. This is not only the case in a whirl, where the speed vectors build circles around the center of the whirl. Even in a uniformly flowing river, in which the speed of the stream decreases from its middle to the banks, a cork will rotate, because the part pointing to the middle is moved faster than the part pointing to the bank. Although all speed vectors are parallel and point in the same direction, such a velocity field of a flowing river contains curls, because the amounts of the parallel speed vectors vary perpendicular to their direction.

If all points of a vector field are known, where the divergence (sources and drains) and curls of the vector field are distinct from zero, and also the strengh of divergences and curls in these points are known, then the complete vector field is determined (Helmholtz theorem) and all field lines can be calculated.

Now we come to answer the first question: Maxwell's equations describe two coupled time-dependent vector fields by giving their sources and curls. Then by these 4 equations after Helmholtz's theorem both fields are fully determined (div E, rot E, div B, rot B).

All four equations are recalled in the following paragraph:

1. Maxwell equation, Coulomb law

Sources of the electrical field are electric charges. Electric field lines start at positive charges and end at negative charges. The divergence of electric field in a point of space is directly proportional to the electrical charge densitiy ρ at that point. The factor in the law is the inverse of the dielectric constant ϵ_0.

$$\text{div } \mathbf{E} = \frac{\rho}{\epsilon_0}$$

2. Maxwell equation

Magnetic fields do not have any sources. There are no magnetic charges so far as we know. Magnetic field lines often are closed curves (ringlike) without starting nor end point. The divergence of magnetic field is thus in every point of space always zero.

$$\text{div } \mathbf{B} = 0$$

3. Maxwell equation, Faraday induction

The curl of an electric field at a point in space is proportional to the change of magnetic field with time at this point, but in opposite direction. For example electric field lines run on a ringlike course around magnetic field lines, when the magnetic field they belong to is changing in time. A test charge running along this field line earns energy (dynamo effect). The proportion factor is the magnetic permeability of space μ_0.

$$\text{rot } \mathbf{E} = -\mu_0 \frac{\partial \mathbf{B}}{\partial t}$$

4. Maxwell equation, Biot-Savart and Ampere

The curl of an magnetic field at a point in space is proportional to the change of electric field E with time in this point or a current density $\mathbf{j}$ in this point. For example magnetic field lines run on a ringlike course around an axis on which a current is flowing or around electric field lines, when the electric field they belong to is changing in time.

$$\text{rot } \mathbf{B} = \mu_0 \mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

A coupling of both fields $\mathbf{E}$ and $\mathbf{B}$ occurs in the 3rd and 4th Maxwell equation for the case of time dependent fields over their curls.

4 The phenomenon behind Maxwell's equations

Maxwell's equations specify for each electric and magnetic field their sources and curls by which the whole fields are given via Helmholtz's theorem. But as *basic mathematical laws*, Maxwell's equations do not say which quantity is a cause and which is a consequence (*they do not give a causal connection between the quantities*).

4.1 Electric and magnetic field are one dual entity

The cause for all electric and magnetic fields are always charges and currents (charges at rest and charges in motion).

A basic statement is included in the 3rd Maxwell's equation ($\mathrm{rot}\mathbf{E} = -\mu_0 \frac{\partial B}{\partial t}$) declaring *two simultanously occuring effects always as equal* (noncausal). Knowing that a constant circular current has a constant magnetic field (elementary magnet), it says that a time-varying current has not only a time-varying magnetic field but also an electric curl field and both effects are equal simultanously after Maxwell's 3rd equation [10].

The coupling of the fields E and B by Maxwell's equations means, that both appear as a dual entity. Therefore both are combined as parts of *one electromagnetic field with six components* (3 electric and 3 magnetic) and instead of electricity and magnetism we speak about electromagnetism (remark: now it is clear, that the statement induction or time-variable magnetic fields cause an electric field or vice versa is *wrong*, both fields occur simultanously caused by moving charges).

4.2 Electromagnetic field from potentials A_1 to A_4 derivable

From the 2nd and 4th Maxwell's equations follows that both fields can be derived from potentials. Because there are no sources for magnetic fields, from the 2nd Maxwell's equation follows that the magnetic field **B** can be written as rotation of another vector field $\mathbf{A} = (A_1, A_2, A_3)$ (as constant of integration there could be a scalar potential which gradient does not contribute here):

$$\mathrm{div}\mathbf{B} = 0 \iff \mathbf{B} = \mathrm{rot}\mathbf{A} \tag{1}$$

The three components of vector **A** depend on 3 coordinates **r** and time t and can be written as three scalar potentials:

$$A_1(\mathbf{r}, t) \;\; A_2(\mathbf{r}, t) \;\; A_3(\mathbf{r}, t)$$

The *vector potential* **A** can be used in the 3rd Maxwell equation and we get:

$$0 \;=\; \mathrm{rot}\mathbf{E} + \mu_0 \frac{\partial}{\partial t}\mathrm{rot}\mathbf{A} \;=\; \mathrm{rot}\left(\mathbf{E} + \mu_0 \frac{\partial \mathbf{A}}{\partial t}\right)$$

Because the term in the brackets has a curl of zero, it can be written as a gradient of a scalar potential φ :

$$\mathbf{E} = -\mu_0 \frac{\partial \mathbf{A}}{\partial t} + \mathrm{grad}\varphi(\mathbf{r}, t) \qquad (2)$$

Thus it is possible to calculate **B** and **E** also from **A** and φ.

The electromagnetic potentials have been introduced here (in classical theory) formally as helping quantities, but *in other theories they are real neccessary fields with physical meaning* (e.g. Schrödinger-equation, QED, Proca-equation).

4.3 Equivalence to wave equation

Through choice of potentials Maxwell's equations can be fulfilled. By insertion into the 4th Maxwell's equation we get a *wave equation* for the vector potential.

$$\mathrm{rot}\ \mathbf{B} = \mathrm{rot}\ \mathrm{rot}\mathbf{A} = \mu_0 \mathbf{j} - \mu_0 \epsilon_0 \mathrm{grad} \frac{\partial \varphi}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

$$\mathrm{rot}\ \mathrm{rot}\mathbf{A} + \mu_0 \epsilon_0 \mathrm{grad} \frac{\partial \varphi}{\partial t} + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu_0 \mathbf{j}$$

$$\mathrm{grad}\ \mathrm{div}\mathbf{A} - \Delta \mathbf{A} + \mu_0 \epsilon_0 \mathrm{grad} \frac{\partial \varphi}{\partial t} + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu_0 \mathbf{j}$$

$$\mathrm{grad} \left(\mathrm{div}\mathbf{A} + \mu_0 \epsilon_0 \mathrm{grad} \frac{\partial \varphi}{\partial t} \right) - \Delta \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu_0 \mathbf{j}$$

Because of degrees of freedom for the choice of the potentials, these can be chosen in such a way that the term in the brackets becomes zero (*Lorentz-gauge*), then

$$\Delta \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j} \qquad (3)$$

This is the wave equation which **A** obeys. By insertion into the 1st Maxwell's equation we obtain the wave equation for the scalar potential:

$$\mathrm{div} \left(-\frac{\partial \mathbf{A}}{\partial t} - \mathrm{grad}\varphi \right) = \frac{\rho}{\epsilon_0}$$

$$-\mathrm{divgrad}\varphi - \frac{\partial}{\partial t} \mathrm{div}\mathbf{A} = \frac{\rho}{\epsilon_0}$$

Because of Lorentz-gauge, div **A** can be substituted:

$$\Delta\varphi - \mu_0\epsilon_0 \frac{\partial^2\varphi}{\partial t^2} = \frac{\rho}{\epsilon_0} \tag{4}$$

This is the wave equation which φ obeys. If on the right-hand sides of the wave equations current density $\mathbf{j}$ and charge density ρ vanish, then we have homogenous wave equations. Solutions are here propagating waves in empty space. The equations (3) and (4) are inhomogenous wave equations.

The *propagation velocity* of the waves is $c = \sqrt{\mu_0\epsilon_0}$ and the square of this being in the wave equations confirmed light as electromagnetic waves [7], [11].

4.4. Wave velocity of light is part of coordinate systems

Even Maxwell pointed out, that these two wave equations prefer a coordinate system at rest, that is a system in which the velocity of light is the same c in every direction. There was no deviation in results for velocity of light (Michelson Morley 1887), even if it's source or observer were moving. This was generalized by Einstein in *special relativity* (1905): it is impossible to find a preferred constant-velocity coordinate system, because physical laws have the same form in every constant-velocity coordinate system. Another coordinate transformation than Galileo's had to be found. The only transformation which leaves light's velocity unchanged when changing from one coordinate system to another is the Lorentz (and Poincare) transformation, which led to *Minkowski metric*, in which space and time are united in a *continuum* and coordinate transformations can be handled like a rotation by an angle $i \cdot \theta$. Therefore 4D distance always contains a space and a time component and every observer measures different spatial and temporal components of distance, depending on their velocity.

In Minkowski space every vector such as velocity and force have to be written with 4 components. So we can combine ρ and $\mathbf{j}$ to a four component current density, with definitions

$$x_4 = ict,\ j_4 = ic\rho,\ A_4 = \frac{i}{c}\varphi \tag{5}$$

both wave equations become one:

$$\frac{\partial^2 A_k}{\partial {x_1}^2} + \frac{\partial^2 A_k}{\partial {x_2}^2} + \frac{\partial^2 A_k}{\partial {x_3}^2} + \frac{\partial^2 A_k}{\partial {x_4}^2} = \mu_0 j_k \quad k = 1, 2, 3, 4 \tag{6}$$

The form of this equation in a constant-velocity system s is unchanged by Lorentz transformation. So if we change to a system s' moving with velocity v relatively to s, we immediateley can write down the equation in s' by substituting all x_i by x'_i, all A_i by A'_i and all j_i by j'_i.

4.5 Field tensor is 4-dimensional rotation of four vector potential

The Faraday tensor or field tensor is given by an equation, with which its 16 components are defined [5] :

$$F_{ik} = \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k} \quad \text{with} \quad i, k = 1, 2, 3, 4 \tag{7}$$

These are the partial derivatives of 4-*dimensional* (4D) *rotation* of A_i. They also occur in 3D rotation and in the gradient, that way the Faraday-tensor contains directly electric and magnetic field components. In Cartesian coorinates they represent equations (1) and (2):

$$\begin{aligned}
B_x &= \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} & \qquad E_x &= ic\left(\frac{\partial A_4}{\partial x_1} - \frac{\partial A_1}{\partial x_4}\right) \\
B_y &= \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} & \qquad E_y &= ic\left(\frac{\partial A_4}{\partial x_2} - \frac{\partial A_2}{\partial x_4}\right) \\
B_z &= \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} & \qquad E_z &= ic\left(\frac{\partial A_4}{\partial x_3} - \frac{\partial A_3}{\partial x_4}\right)
\end{aligned} \tag{8}$$

which were derived from 2nd and 3rd Maxwell's equation. The tensor F_{ik} is the 4D rotation of four vectors A_i. It can be written this way with use of the electromagnetic field quantities:

$$\mathsf{F} = (F_{ik}) = \begin{bmatrix} 0 & B_z & -B_y & -\frac{i}{c}E_x \\ -B_z & 0 & -B_x & -\frac{i}{c}E_y \\ B_y & B_x & 0 & -\frac{i}{c}E_z \\ -\frac{i}{c}E_x & -\frac{i}{c}E_y & -\frac{i}{c}E_z & 0 \end{bmatrix} \tag{9}$$

Here we have found a physical quantity, which unifies electric and magnetic field components into one dual entity: the electromagnetic field tensor F.

With it we can combine the 4 Maxwell's equations into two equations:

$$\left.\begin{array}{l}\mathrm{rot}\mathbf{E} = -\mu_0 \frac{\partial \mathbf{B}}{\partial t} \\ \\ \mathrm{div}\mathbf{B} = 0\end{array}\right\} \rightarrow \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} = 0 \tag{10}$$

This so-called homogenous equation is implied by the definition of F through 2nd and 3rd Maxwell's equation ($\lambda, \mu, \nu = 1..4$). The 1st and 4th Maxwell's equation are combined to the so-called inhomogenous equation:

$$\left.\begin{array}{l}\mathrm{rot}\mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \\ \\ \mathrm{div}\mathbf{E} = \rho/\epsilon_0\end{array}\right\} \rightarrow \quad \frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 j_\nu \tag{11}$$

Now we are close to the answer of the second question: *The phenomenon behind Maxwell's equations can be seen in the field tensor, which is written as a 4x4-matrix* (zero-diagonal, 6 field components) *and built by 4D rotation of four vector potential, which has its basis in the 4 source components.*

4.6 Causal or retarded four vector potential is the solution of the inhomogenous wave equation for the 4 potentials

The solutions for the inhomogenous wave equation in a piont $\mathbf{r}$ at time t is mathematically:

$$A_1 = \mu_0 \int\limits_V \frac{j_1\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}\, d^3r'$$

$$A_2 = \mu_0 \int\limits_V \frac{j_2\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}\, d^3r'$$

$$A_3 = \mu_0 \int\limits_V \frac{j_3\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}\, d^3r'$$

$$A_4 = \frac{i}{c\epsilon_0} \int\limits_V \frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}\, d^3r'$$

In the Integral the main terms are the sources (causes). Because points without sources do not contribute to the integral, the integral has to be carried out only over the areas with sources. The source terms have a time retarding term to take into account that a signal needs time to propagate from position **r'** to position **r**, and also a factor of reciprocal distance. Each component of the four vector potential is directly a result of the sources.

5 Maxwell's equations in a curvature tensor

Riemann presented his concept of geometry in his habilitation lecture. He characterized it by the definition of points as n-tuples of numbers, a space being the set of all points, and a metric space having a distance function between every two points of the space. That way Riemann's geometry is completely based on analysis, in contrast to Euclid's geometry, which starts with descriptive nominal definitions such as a point is something which has no parts, a line is length without thickness, an angle is an inclination between two lines. Riemann distinguished his geometry also from Gauss's theory of curved 2D surfaces, which need a 3D euclidean space for their definition, and from those non-euclidean geometries of Bolyai and Lobatchevski.

Following Riemann's general concept we start with points P as n-tuples of a n-dimensional manifold

$$P = x = (x_1, x_2, ..., x_n) \tag{12}$$

forming a curve $x(t)$ parameterized by t from t_0 to t_1, which runs from starting point $P_0 = x(t_0)$ to end point $P_1 = x(t_1)$:

$$x(t) = (x_1(t), x_2(t), ..., x_n(t)) \tag{13}$$

Between two neighbouring points with infinitesimal *difference in coordinates*

$$dx = (dx_1, dx_2, ..., dx_n) \tag{14}$$

shall be defined the infinitesimal *difference in distance*:

$$ds = F(x, dx) \tag{15}$$

that means the distance depends on position $x(t)$ and the change of coordinates $dx_i(t)$ in the direction of the chosen path.

Riemann first investigated distance functions which were made from positive definite quadratic forms. He supposed the next easy case to investigate would be positive definite differential terms of 4th degree. But historically physics demanded another agenda: special relativity led to indefinite quadratic forms, which are differences between terms of second degree, as for example $x^2 - (ct)^2$, where all points lying on the 45° diagonal (2D light cone) have zero distance between each other (distance zero between two points does not imply that they have the same coordinates).

The distance as we know it from daily life shall be independent of the direction in which the curve is run through and should be summed up linearly:

$$F(x, k \cdot dx) = |k| F(x, dx) \tag{16}$$

To be able to define the function 'amount of k' as $|k| = +\sqrt{k^2}$ and for including euclidean space with $ds^2 = {dx_1}^2 + {dx_2}^2 + {dx_2}^2$, it is neccessary, that F^2 has to be a *homogenous function of* 2nd degree:

$$F^2(x, k \cdot dx) = k^2 F(x, dx) \tag{17}$$

and therefore F^2 is a quadratic differential form:

$$F^2(x, dx) = g_{ij}(x)\ dx_i dx_j \tag{18}$$

This expression has the special name *fundamental quadratic form* or *metric form*. The length of the curve is defined as (summed up via integration):

$$s = \int_{t_0}^{t_1} +\sqrt{F^2(x, \frac{dx}{dt})} dt \tag{19}$$

5.1 Metric Tensor and curvature of space

For interpretation of the terms $g_{ij}\ dx_i dx_j$ we now take a closer look at the coordinate line $u_i(t')$ through a point $x(t)$, which we get by fixing all coodinates of that point except one. For the coordinates with fixed t we write instead of $x_i(t)$ simply x_i:

$$u_1(t') = (x_1(t'), x_2, ..., x_n)$$

$$u_2(t') = (x_1, x_2(t'), ..., x_n)$$

$$...$$

$$u_n(t') = (x_1, x_2, ..., x_n(t'))$$

Denoting the distance ds_i between two neighbouring points on the coordinate line u_i, which have coordinate difference du_i

$$ds_i = +\sqrt{F^2(x, du_i)} \tag{20}$$

we get (because here for all $j \neq i$ we have vanishing $du_j = 0$):

$$ds_i = +\sqrt{g_{ii}(du_i)^2} = +\sqrt{g_{ii}}\ du_i \tag{21}$$

With definition:

$$d\mathbf{r} = (ds_1, ds_2, ..., ds_n) \tag{22}$$

then we get

$$\frac{\partial \mathbf{r}}{\partial u_i} = \frac{d\mathbf{s}_i}{du_i} = \sqrt{g_{ii}}\ \mathbf{T}_i = \mathbf{a}_i \tag{23}$$

where $\mathbf{T}_i$ is the *unity tangent vector* for u_i-curve in point $x(t)$. Herein we have shortened $\sqrt{g_{ii}}\ \mathbf{T}_i$ as $\mathbf{a}_i$ by a definition.

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1}du_1 + \frac{\partial \mathbf{r}}{\partial u_2}du_2 + \ ...\ + \frac{\partial \mathbf{r}}{\partial u_n}du_n = \mathbf{a}_1 du_1 + \mathbf{a}_2 du_2 + \ ...\ + \mathbf{a}_n du_n \tag{24}$$

We get the square of length of $d\mathbf{r}$ by [12]:

$$\begin{aligned} ds^2 = d\mathbf{r} \cdot d\mathbf{r} = &\ \mathbf{a}_1 \cdot \mathbf{a}_1 {du_1}^2 + \mathbf{a}_1 \cdot \mathbf{a}_2 du_1 du_2 + \mathbf{a}_1 \cdot \mathbf{a}_3 du_1 du_3 \\ &+\mathbf{a}_2 \cdot \mathbf{a}_1 du_2 du_1 + \mathbf{a}_2 \cdot \mathbf{a}_2 {du_2}^2 + \mathbf{a}_2 \cdot \mathbf{a}_3 du_2 du_3 \\ &+\mathbf{a}_3 \cdot \mathbf{a}_1 du_3 du_1 + \mathbf{a}_3 \cdot \mathbf{a}_2 du_3 du_2 + \mathbf{a}_3 \cdot \mathbf{a}_3 {du_3}^2 \\ &= \sum_{p=1}^{n} \sum_{q=1}^{n} g_{pq} du_p du_q \text{ where } g_{pq} = \mathbf{a}_p \cdot \mathbf{a}_q \end{aligned}$$

Now we do another decomposition of d$\mathbf{r}$ with *untity normal vectors* $\mathbf{N}_i$, which are orthogonal to a coordinate surface through $x(t)$ defined by having the same u_i = const everywhere. We get such orthogonal vectors by building the gradient of u_i. Because we can not ad hoc compute the length in this direction, we define new g^{pq} (other values indicated by upper index).

$$d\mathbf{r} = \nabla u_1 du_1 + \nabla u_2 du_2 + \ ...\ + \nabla u_n du_n = \mathbf{b}_1 du_1 + \mathbf{b}_2 du_2 + \ ...\ + \mathbf{b}_n du_n \tag{25}$$

We get the length of $d\mathbf{r}$ again by squaring:

$$\begin{aligned} ds^2 = d\mathbf{r} \cdot d\mathbf{r} = &\ \mathbf{b}_1 \cdot \mathbf{b}_1 {du_1}^2 + \mathbf{b}_1 \cdot \mathbf{b}_2 du_1 du_2 + \mathbf{b}_1 \cdot \mathbf{b}_3 du_1 du_3 \\ &+\mathbf{b}_2 \cdot \mathbf{b}_1 du_2 du_1 + \mathbf{b}_2 \cdot \mathbf{b}_2 {du_2}^2 + \mathbf{b}_2 \cdot \mathbf{b}_3 du_2 du_3 \\ &+\mathbf{b}_3 \cdot \mathbf{b}_1 du_3 du_1 + \mathbf{b}_3 \cdot \mathbf{b}_2 du_3 du_2 + \mathbf{b}_3 \cdot \mathbf{b}_3 {du_3}^2 \\ &= \sum_{p=1}^{n} \sum_{q=1}^{n} g^{pq} du_p du_q \text{ where } g^{pq} = \mathbf{b}_p \cdot \mathbf{b}_q \end{aligned}$$

Both basis vector systems are so called reciprocal systems, e.g. their scalar product is 1 :

$$\begin{gathered} \mathbf{a}_1 \cdot \mathbf{b}_1 = 1 \\ \mathbf{a}_2 \cdot \mathbf{b}_2 = 1 \\ ... \\ \mathbf{a}_n \cdot \mathbf{b}_n = 1 \\ \sum_{p=1}^{n} \sum_{q=1}^{n} g_{pq} g^{pq} = 1 \\ \text{or with Einstein's sumconvention } g_{pq} g^{pq} = 1 \end{gathered}$$

With it the g^{pq} can be calculated (inverse matrix):

$$(g^{pq}) = (g_{pq})^{-1}$$

The reciprocity of the basis vector systems on which the metric tensors are defined have the following advantage: decomposing a vector $\mathbf{A}$ after $\mathbf{a}_i$ and $\mathbf{b}_i$ we get

$$\mathbf{A} = A^1\mathbf{b}_1 + A^2\mathbf{b}_2 + ... + A^n\mathbf{b}_n$$
$$\mathbf{A} = A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + ... + A_n\mathbf{a}_n$$

and the amount or strenght of the vector can be computed as

$$\mid \mathbf{A} \mid^2 = \sum_{p=1}^{n} A_p A^p$$

The g_{pq} and g^{pq} can both be written as a matrices. Their quadratic forms then can be generated by matrix product like $(dx_i) \times (g^{pq}) \times (dx_j)$. Both metric tensors belonging to the two basis vector systems describe the same space but are different in shape, i.e. the elements in rows and columns are different. Thus the shape of a metric tensor depends on the chosen coordinate system. Looking only on the $g_{pq}(x)$ we can not decide whether the space is flat or curved. But if it is possible to do a coordinate transformation to a Cartesian form, then the space is flat:

$$ds^2 = dx^2 + dy^2 + dz^2$$

The metric of the same space is in cylinder coordinates:

$$ds^2 = d\rho^2 + r^2 d\varphi^2 + dz^2$$

which has also only diagonal non-zero elements, but they are not constant and it can not be seen, whether the space is flat or not. The metric of the surface of a sphere can not be transformed to a Cartesian form:

$$ds^2 = R^2 \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2 \right)$$

Independently from a special coordinate transformation, another tensor can be build from the metric tensor, namely the curvature tensor, which is exactly then zero when the space is flat. For calculation of the curvature tensor one has to evaluate so-called *Christoffel symbols* [13], which are terms built from g^{pq} and derivatives of the g_{pq} (with Einstein's summation convention):

$$\Gamma^{\kappa}_{\lambda\mu} = \frac{g^{\kappa\nu}}{2} \left(\frac{\partial g_{\mu\nu}}{\partial u^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial u^{\mu}} - \frac{\partial g_{\lambda\mu}}{\partial u^{\nu}} \right) \tag{26}$$

Because of the derivatives in the expression, often many terms are zero and the calculations become less complicated. This is also important for a special physical application: if the first term of the sum in the brackets of a Christoffel symbol is zero, then the second and third are similar to the rotation operator in Maxwell's equations! Kaluza's idea was to use an additional dimension to the four used in Minkowski space, and to place electromagnetism in those additional Christoffel symbols. Note – Christoffel symbols are not tensors.

The *curvature tensor* is built from Christoffel symbols and their derivatives [13]:

$$R^{m}{}_{ikp} = \frac{\Gamma g^{m}_{ik}}{\partial u^{p}} - \frac{\Gamma g^{m}_{ip}}{\partial u^{k}} + \Gamma^{r}_{ik}\Gamma^{m}_{rp} - \Gamma^{r}_{ip}\Gamma^{m}_{rk} \tag{27}$$

When this tensor is vanishing (zero matrix), then space is flat:

$$R^{m}{}_{ikp} = 0 \ for\ a\ space\ without\ curvature.$$

From the curvature tensor another famous tensor can be built: the *Ricci tensor*, which forms from the above tensor, when two indices are identical. Each element of the new tensor is a sum of a diagonal, a so called trace:

$$R_{mn} = R^{r}{}_{mrn} = g^{kr}\ R_{kmrn}$$

The trace of the Ricci tensor is called *curvature scalar*:

$$R = R^{m}{}_{m}$$

The Ricci tensor includes second derivatives of the metric coefficients g_{pq}. Because these are parts of an indefinite metric, parts of the Ricci tensor can include wave equations!

5.2 Choice of a specific metric tensor

Following Kaluza we move on to a special space with a special metric tensor, which has the first four dimensions in accordance with Minkowski space and additional a fifth linear independent basis vector.

The orthogonal coordinate lines in a system s are x, y, z, $\tau = ict$ (Minkowski-space) and $\omega = \sqrt{\varepsilon}w$ with $\varepsilon = \pm 1$. The basis vectors for these lines are $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, $\mathbf{l}$, $\mathbf{m}$, all orthogonal to each other and of amount 1, giving radius vector $\mathbf{r} = \mathrm{x}\ \mathbf{i} + \mathrm{y}\ \mathbf{j} + \mathrm{z}\ \mathbf{k} + \tau\ \mathbf{l} + \omega\ \mathbf{m}$. Usually coordinate lines are real, but here we hide the signature of the metric as in Minkowski-space giving iw or w for ω. A second coordinate system s' in this 5-dimensional space, which is reciprocal to the first, has coordinate lines x', y', z', $\tau' = ict'$, the same basis vectors from Minkowski space, but for u', the fifth coordinate line, a basis vector $\mathbf{A}$ inclined to Minkowski hypersurface.

contravariant System

A (very small) radius vector $\mathbf{r}$ can be written in s as:

$\mathbf{r} = \mathrm{x}'\ \mathbf{i} + \mathrm{y}'\ \mathbf{j} + \mathrm{z}'\ \mathbf{k} + \tau'\ \mathbf{l} + \mathrm{u}'\ \mathbf{A}$

The coordinate transformation for $\mathbf{r}$ from inclined system s' back to fully orthogonal system s is:

$$\begin{aligned}
x &= x' + u'A_x \\
y &= y' + u'A_y \\
z &= z' + u'A_z \\
\tau &= \tau' + u'A_\tau \\
\omega &= u'A_\omega \qquad (\text{ for } \varepsilon = -1: \quad A_\omega = iA_w \quad A_\omega^2 = -A_w \quad A_\omega^4 = A_w^4\ )
\end{aligned}$$

An infinitesimal distance (line element) transforms from s' to s as:

$$\begin{aligned}
d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial x'}\,\mathrm{dx}' + \frac{\partial \mathbf{r}}{\partial \mathrm{y}'}\,\mathrm{dy}' + \frac{\partial \mathbf{r}}{\partial \mathrm{z}'}\,\mathrm{dz}' + \frac{\partial \mathbf{r}}{\partial \tau'}\,\mathrm{d}\tau' + \frac{\partial \mathbf{r}}{\partial \mathrm{u}'}\,\mathrm{du}' \\
&= \mathbf{i}\,\mathrm{dx}' + \mathbf{j}\,\mathrm{dy}' + \mathbf{k}\,\mathrm{dz}' + \mathbf{l}\,\mathrm{i}\mathrm{d}\tau' + (\mathbf{i}\,\mathrm{A_x} + \mathbf{j}\,\mathrm{A_y} + \mathbf{k}\,\mathrm{A_z} + \mathbf{l}\,\mathrm{A}_\tau + \mathbf{m}\,\mathrm{A}_\omega)\,\mathrm{du}'
\end{aligned}$$

The square length of the line element is:

$$\begin{aligned}
d\mathbf{r}\cdot \mathrm{d}\mathbf{r} &= (dx')^2 + 2A_x dx'du' + (dy')^2 + 2A_y dy'du' + (dz')^2 + 2A_z dz'du' \\
&\quad +(d\tau')^2 + 2A_\tau d\tau' du' + \underbrace{(A_x^2 + A_y^2 + A_z^2 + A_\tau^2 + A_\omega^2)}_{A^2}(du')^2
\end{aligned}$$

which is $(dx_i') \cdot (g_{AB}) \cdot (dx_i')$ with metric tensor (g_{AB}):

$$(g_{AB}) = \begin{pmatrix} 1 & 0 & 0 & 0 & A_x \\ 0 & 1 & 0 & 0 & A_y \\ 0 & 0 & 1 & 0 & A_z \\ 0 & 0 & 0 & 1 & A_\tau \\ A_x & A_y & A_z & A_\tau & A^2 \end{pmatrix} \qquad \det g_{AB} = A^2 - A_x^2 - A_y^2 - A_z^2 - A_\tau^2 = A_\omega^2$$

covariant System

coordinate transformation from orthogonal system s to inclined system s':

$$\begin{aligned}
x'(x,y,z,\tau,\omega) &= x - \frac{A_x}{A_\omega}\omega \\
y'(x,y,z,\tau,\omega) &= y - \frac{A_y}{A_\omega}\omega \\
z'(x,y,z,\tau,\omega) &= z - \frac{A_z}{A_\omega}\omega \\
\tau'(x,y,z,\tau,\omega) &= \tau - \frac{A_\tau}{A_\omega}\omega \\
u'(x,y,z,\tau,\omega) &= \frac{1}{A_\omega}\omega
\end{aligned}$$

gradient of coordinate: $\nabla x' = \frac{\partial x'}{\partial x} + \frac{\partial x'}{\partial y} + \frac{\partial x'}{\partial z} + \frac{\partial x'}{\partial \tau} + \frac{\partial x'}{\partial \omega}$

A line element in reciprocal system s':

$$\begin{aligned} d\mathbf{r} &= \nabla x' dx + \nabla y' dy + \nabla z' dz + \nabla \tau' d\tau + \nabla u' d\omega = \left(\mathbf{i} - \frac{A_x}{A_\omega}\mathbf{m}\right) dx \\ &+ \left(\mathbf{j} - \frac{A_y}{A_\omega}\mathbf{m}\right) dy + \left(\mathbf{k} - \frac{A_z}{A_\omega}\mathbf{m}\right) dz + \left(\mathbf{l} - \frac{A_\tau}{A_\omega}\mathbf{m}\right) d\tau + \frac{1}{A_\omega}\mathbf{m}\, d\omega \end{aligned}$$

and the resulting metric tensor (g^{AB})

$$(g^{AB}) = \begin{pmatrix} 1 + \frac{A_x^2}{A_\omega^2} & \frac{A_x A_y}{A_\omega^2} & \frac{A_x A_z}{A_\omega^2} & \frac{A_x A_\tau}{A_\omega^2} & \frac{-A_x}{A_\omega^2} \\ \frac{A_y A_x}{A_\omega^2} & 1 + \frac{A_y^2}{A_\omega^2} & \frac{A_y A_z}{A_\omega^2} & \frac{A_y A_\tau}{A_\omega^2} & \frac{-A_y}{A_\omega^2} \\ \frac{A_z A_x}{A_\omega^2} & \frac{A_z A_y}{A_\omega^2} & 1 + \frac{A_z^2}{A_\omega^2} & \frac{A_z A_\tau}{A_\omega^2} & \frac{-A_z}{A_\omega^2} \\ \frac{A_\tau A_x}{A_\omega^2} & \frac{A_\tau A_y}{A_\omega^2} & \frac{A_\tau A_z}{A_\omega^2} & 1 + \frac{A_\tau^2}{A_\omega^2} & \frac{-A_\tau}{A_\omega^2} \\ \frac{-A_x}{A_\omega^2} & \frac{-A_y}{A_\omega^2} & \frac{-A_z}{A_\omega^2} & \frac{-A_\tau}{A_\omega^2} & \frac{1}{A_\omega^2} \end{pmatrix}$$

5.3 Calculating $\mathbf{R_{55}}$ (A_w constant)

For a first and easy calculation let us start with basis vector products g^{AB} depending only on coordinates in Minkowski space, so they are independent of the fifth coordinate (constant along a fiber). Then all derivatives after the fifth coordinate become zero. Because all g^{ij} with i, j = 1..4 are constants of either one or zero, also their derivatives vanish. Only the five g^{A5} are left for derivation. In addition as another simplification A_w shall be constant ($A_w = 1$) and so all derivatives of A_w also are zero. Now we have 4 varying values left: A_x, A_y, A_z, A_t, which are tangent to Minkowski space. A_w is perpendicular to Minkowski hypersurface and can be seen as height or thickness of a layer.

We now compute the Christoffel symbols from the reciprocal metric tensors g^{AB} and g_{AB}. Small Greek indices denote numbers from 1 to 4 (Minkowski-space) and big latin letters numbers from 1 to 5 (5D space):

$$\Gamma^{A}_{BC} = \sum_{D=1}^{5} \frac{g^{AD}}{2} \left(\frac{\partial g_{CD}}{\partial u^B} + \frac{\partial g_{BD}}{\partial u^C} - \frac{\partial g_{CB}}{\partial u^D} \right) \tag{28}$$

We want to investigate the following Christoffel symbols: Γ^5_{55}, Γ^α_{55} und $\Gamma^\alpha_{\beta 5} = \Gamma^\alpha_{5\beta}$. Lets start with Γ^5_{55}. The first two terms in the brackets are derivations after u^5, which shall be zero by assumption (no u^5 dependency). All derivatives of A_w and therefore of g_{55} vanish also:

$$\begin{aligned}\Gamma^5_{55} &= \sum_{D=1}^{5} \frac{g^{5D}}{2}\left(\frac{\partial g_{5D}}{\partial u^5} + \frac{\partial g_{D5}}{\partial u^5} - \frac{\partial g_{55}}{\partial u^D}\right) && (29)\\ &= 0 && (30)\end{aligned}$$

The same applies to Γ^α_{55}:

$$\begin{aligned}\Gamma^\alpha_{55} &= \sum_{D=1}^{5} \frac{g^{\alpha D}}{2}\left(\frac{\partial g_{5D}}{\partial u^5} + \frac{\partial g_{D5}}{\partial u^5} - \frac{\partial g_{55}}{\partial u^D}\right) && (31)\\ &= 0 && (32)\end{aligned}$$

For the next Christoffel symbol $\Gamma^\alpha_{\beta 5}$ we find that the summand in the middle is zero and also the term for $D = 5$. Because the first four dimensions belong to Minkowski space, only the diagonal element $g^{\alpha\alpha} = 1$ remains from the row with $g^{\alpha\delta}$:

$$\begin{aligned}\Gamma^\alpha_{\beta 5} &= \sum_{D=1}^{5} \frac{g^{\alpha D}}{2}\left(\frac{\partial g_{5D}}{\partial u^\beta} + \frac{\partial g_{\beta D}}{\partial u^5} - \frac{\partial g_{\beta 5}}{\partial u^D}\right) && (33)\\ &= \sum_{\delta=1}^{4} \frac{g^{\alpha\delta}}{2}\left(\frac{\partial g_{5\delta}}{\partial u^\beta} + 0 - \frac{\partial g_{\beta 5}}{\partial u^\delta}\right) && (34)\\ &= \frac{g^{\alpha\alpha}}{2}\left(\frac{\partial(\frac{-A_\alpha}{A_\omega^2})}{\partial u^\beta} - \frac{\partial(\frac{-A_\beta}{A_\omega^2})}{\partial u^\alpha}\right) && (35)\\ &= -\frac{1}{2\varepsilon A_w^2}\left(\frac{\partial A_\alpha}{\partial u^\beta} - \frac{\partial A_\beta}{\partial u^\alpha}\right) && (36)\\ &= -\frac{1}{2}\,\varepsilon\,\phi^2 F_{\alpha\beta} && (37)\end{aligned}$$

In the last step we have substituted the brackets with the field tensor component and also $\phi = \frac{1}{A_w}$.

If we now calculate the Ricci-Tensor R_{AB} [13]:

$$R_{AB} = R^D{}_{ADB} = \sum_{D=1}^{5} \frac{\partial \Gamma^D_{AD}}{\partial u^B} - \sum_{D=1}^{5} \frac{\partial \Gamma^D_{AB}}{\partial u^D} + \sum_{D=1}^{5}\sum_{E=1}^{5}\left(\Gamma^E_{AD}\Gamma^D_{EB} - \Gamma^E_{AB}\Gamma^D_{ED}\right) \quad (38)$$

with $A = B = 5$. So the first term vanishes because derviation after u^5 and the terms directly after the minus sign are zero ($\Gamma^\alpha_{55} = 0$). It remains only the third summand, where again for $D = 5$ and $E = 5$ we get zero:

$$
\begin{aligned}
R_{55} &= \sum_{D=1}^{5} \frac{\partial \Gamma^{D}_{5D}}{\partial u^{5}} - \sum_{D=1}^{5} \frac{\partial \Gamma^{D}_{55}}{\partial u^{D}} + \sum_{D=1}^{5} \sum_{E=1}^{5} \Gamma^{E}_{5D} \Gamma^{D}_{E5} - \sum_{D=1}^{5} \sum_{E=1}^{5} \Gamma^{E}_{55} \Gamma^{D}_{ED} && (39) \\
&= \sum_{\delta=1}^{4} \sum_{\epsilon=1}^{5} \Gamma^{\epsilon}_{5\delta} \Gamma^{\delta}_{\epsilon 5} && (40) \\
&= \sum_{\delta=1}^{4} \sum_{\epsilon=1}^{5} (-\frac{1}{2}) \varepsilon \phi^{2} F_{\epsilon\delta} (-\frac{1}{2}) \varepsilon \phi^{2} F_{\delta\epsilon} && (41) \\
&= \sum_{\delta=1}^{4} \sum_{\epsilon=1}^{5} \frac{1}{4} \varepsilon^{2} \phi^{4} F_{\epsilon\delta} F_{\epsilon\delta} && (42) \\
&= \frac{1}{4} \phi^{4} (\mathbf{E} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{B} + \mathbf{S} \cdot \mathbf{S}) && (43)
\end{aligned}
$$

R_{55} corresponds with the sum of static electric energy, static magnetic energy and electromagnetic radiation energy [14].

5.4 Calculating $\mathbf{R}_{5\mu}$ (A_w constant)

For calculation of $R_{5\mu}$ we need the Christoffel symbols from the preceeding section and $\Gamma^{\alpha}_{\beta\gamma}$. In its sum from the first four summands only the term with the diagonal element $g^{\alpha\alpha}$ remains, which is non-zero as $g^{\alpha 5}$ is. Inserting, application of product rule and omitting terms which sum is zero leads to the final expression:

$$
\begin{aligned}
\Gamma^{\alpha}_{\beta\gamma} &= \sum_{D=1}^{5} \frac{g^{\alpha D}}{2} \left(\frac{\partial g_{\gamma D}}{\partial u^{\beta}} + \frac{\partial g_{\beta D}}{\partial u^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial u^{D}} \right) && (44) \\
&= \frac{g^{\alpha\alpha}}{2} \left(\frac{\partial g_{\gamma\alpha}}{\partial u^{\beta}} + \frac{\partial g_{\beta\alpha}}{\partial u^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial u^{\alpha}} \right) && (45) \\
&+ \frac{g^{\alpha 5}}{2} \left(\frac{\partial g_{\gamma 5}}{\partial u^{\beta}} + \frac{\partial g_{\beta 5}}{\partial u^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial u^{5}} \right) \\
&= \frac{1}{2} \frac{1}{A_{\omega}^{2}} \left(\frac{\partial (A_{\gamma} A_{\alpha})}{\partial u^{\beta}} + \frac{\partial (A_{\beta} A_{\alpha})}{\partial u^{\gamma}} - \frac{\partial (A_{\beta} A_{\gamma})}{\partial u^{\alpha}} \right) && (46) \\
&+ \frac{1}{2} \frac{1}{A_{\omega}^{2}} \left(-A^{\alpha} \frac{\partial A_{\gamma}}{\partial u^{\beta}} - A^{\alpha} \frac{\partial A_{\beta}}{\partial u^{\gamma}} \right) \\
&= \frac{1}{2} \frac{1}{A_{\omega}^{2}} \left(\frac{\partial A_{\alpha}}{\partial u^{\beta}} + \frac{\partial A_{\alpha}}{\partial u^{\gamma}} - \frac{\partial (A_{\beta} A_{\gamma})}{\partial u^{\alpha}} \right) && (47) \\
&= \frac{1}{2} \varepsilon \phi^{2} \left(A_{\gamma} F_{\alpha\beta} + A_{\beta} F_{\alpha\gamma} \right) && (48)
\end{aligned}
$$

Now we have all neccessary Christoffel symbols to write down the expressions for the 4 values $R_{5\mu}$. In the first sum the term for $D = 5$ vanishes, because Γ^5_{55} is zero and in the second sum because dervivatives after u^5 are zero. The first sum vanishes completely because $\Gamma^\delta_{5\delta}$ is zero for all δ. The double sum over products of Christoffel symbols is split into terms with and without $D, E = 5$:

$$
\begin{aligned}
R_{5\mu} &= \sum_\delta \frac{\partial \Gamma^\delta_{5\delta}}{\partial u^\mu} - \sum_\delta \frac{\partial \Gamma^\delta_{5\mu}}{\partial u^\delta} + \sum_\epsilon \sum_\delta \left(\Gamma^\epsilon_{5\delta}\Gamma^\delta_{\epsilon\mu} - \Gamma^\epsilon_{5\mu}\Gamma^\delta_{\epsilon\delta}\right) \\
&\quad + \sum_\epsilon \left(\Gamma^\epsilon_{55}\Gamma^5_{\epsilon\mu} - \Gamma^\epsilon_{5\mu}\Gamma^5_{\epsilon 5}\right) \\
&\quad + \sum_\delta \left(\Gamma^5_{5\delta}\Gamma^\delta_{5\mu} - \Gamma^5_{5\mu}\Gamma^\delta_{5\delta}\right) \\
&\quad + \Gamma^5_{55}\Gamma^5_{5\mu} - \Gamma^5_{5\mu}\Gamma^5_{55} && (49) \\
&= - \sum_\delta \frac{\partial \Gamma^\delta_{5\mu}}{\partial u^\delta} + \sum_\epsilon \sum_\delta \left(\Gamma^\epsilon_{5\delta}\Gamma^\delta_{\epsilon\mu} - \Gamma^\epsilon_{5\mu}\Gamma^\delta_{\epsilon\delta}\right) && (50)
\end{aligned}
$$

From the eight products of Christoffel symbols only the first two remain. Product seven and eight contain Γ^5_{55} which is zero, product six contains $\Gamma^\delta_{5\delta}$ which is zero for all δ and product three contains Γ^ϵ_{55}, also zero for all ϵ. Renaming δ to ϵ in product five makes it the negative of product four and their sum becomes zero. For simplification we split $R_{5\mu}$ into single sum and double sum:

$$R_{5\mu} = S_1 + S_2 \tag{51}$$

We now insert our results, first in S_1:

$$S_1 = \frac{1}{2}\varepsilon\phi^2 \sum_\delta \frac{\partial F^{\delta\mu}}{\partial u^\delta} \tag{52}$$

which is the left side of Maxwell's source equations multiplied by $\phi^2/2$. Now we insert into S_2 (for the second product δ and ϵ exchange, $F_{\delta\delta} = 0$ for all δ):

$$
\begin{aligned}
S_2 &= \sum_\epsilon \sum_\delta \frac{1}{2}\varepsilon\phi^2 F_{\epsilon\delta} \cdot \frac{1}{2}\varepsilon\phi^2 \left(A_\mu F_{\delta\epsilon} + A_\epsilon F_{\delta\mu}\right) && (53) \\
&\quad - \frac{1}{2}\varepsilon\phi^2 F_{\epsilon\mu} \cdot \frac{1}{2}\varepsilon\phi^2 \left(A_\delta F_{\delta\epsilon} + A_\epsilon F_{\delta\delta}\right) && (54) \\
&= \sum_\epsilon \sum_\delta \frac{1}{4}\varepsilon^2\phi^4 \left[F_{\epsilon\delta}\left(A_\mu F_{\delta\epsilon} + A_\epsilon F_{\delta\mu}\right) - F_{\delta\mu}\ A_\epsilon F_{\epsilon\delta}\right] && (55) \\
&= A_\mu \sum_\epsilon \sum_\delta \frac{1}{4}\phi^4 F_{\epsilon\delta} F_{\epsilon\delta} && (56) \\
&= A_\mu\ R_{55} && (57)
\end{aligned}
$$

Now we finished the calcualation of the 5th row of the Ricci tensor given by the special metric of section 5.2 and in detail we discuss that in the next section.

6 Conclusion

Following the idea of embedding Minkowski space and Maxwell's equation in a 5D Riemannian space we determined the special metric of section 5.2 by the ten elements of the pseudo euclidean metric (because of symmetry of metric tensors the elements in the upper right and lower left are the same), by the four A_1 to A_4 and by the fifteenth element, which was set constant. The four A_μ were assumed to be a solution of a homogenous wave equation, so we know that the expression denoted as S_1, which is the left hand side of Maxwell's source equation, becomes zero (locally no sources) and we calculated the curvature of the 5D space as $R_{5\mu} = A_\mu R_{55}$ where R_{55} corresponds to local energy. So far the embeding was successful.

This result means that Minkowski space is external curved in a special manner, like a sheet of paper rolled up to a cylinder having still no internal curvature. To roll up one axis (dimension) to a circle one needs two dimensions. So in four spatial dimensions both axes of a sheet of paper can be bend externally without causing internal curvature (similar to a torus), and in six spatial dimensions all three axes of an euclidean cube can be bend to circles externally without changing the internal euclidean metric [15].

Here it has to be emphasized that this embedding is pure mathematics. In physical theories all 25 components of the Ricci tensor often are set to zero: $R_{AB} = 0$. Such a theory needs a curved 4D space for gravitation and variable fifteenth element (A_w) to compensate the electromagnetic energy in R_{55} to zero. As described by Ferrari [16] three four-dimensional equations can be derived from the five-dimensional equation $R_{AB} = 0$: Maxwell-Einstein tensor equation (gravity field with electromagntic energy as source), Maxwell's vector equations of vacuum and a scalar wave equation for the fifteenth element:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R &= -\frac{k^2\phi^2}{2}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{g_{\mu\nu}}{4}F_{\alpha\beta}F^{\alpha\beta}\right) \\ &\quad - \frac{1}{\phi}\quad \left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla_\alpha\nabla^\alpha\phi\right) \qquad (58) \\ \nabla_\alpha\left(\phi^3 F^{\alpha\mu}\right) &= 0 \qquad (59) \\ \nabla_\mu\nabla^\mu\phi &= \frac{k^2\phi^3}{4}F_{\mu\nu}F^{\mu\nu} \qquad (60) \end{aligned}$$

In the first of Ferrari's formulas the factor $k^2\phi^2/2$ plays the role of the gravity constant $8\pi\gamma$. It is an open question whether the feature of a modifiable strenght of gravity (inertia) has any meaning in reality. If this is the case, another question will be whether fields can reach the strength neccessary to predict measurable effects.

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